M.Sc. - I (Mathematics) (New CBCS Pattern) Semester-II

PSCMTH07 - Lebesgue Measure Theory

P. Pages : 2 **GUG/S/25/13747**

Time : Three Hours Max. Marks : 100

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Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let A be any set and $E_1, E_2 \ldots . E_n$ a finite sequence of disjoint measurable sets. Then prove that $m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$ **10**

b) Prove that the collection M (the set of all measurable sets) of measurable sets is a σ-algebra. **10**

OR

c) Let $\langle E_i \rangle$ be a sequence of measurable sets. Then prove that $m(\cup E_i) \le \sum mE_i$. Also, prove that, if the sets E_n are pairwise disjoint, then $m(\cup E_i) = \sum mE_i$. **10**

d) If $\langle f_n \rangle$ is a sequence of measurable functions with the same domain of definition, then prove that the function $\sup\{f_1, \ldots, f_n\}, \inf\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, and $\overline{\lim}\, f_n$ are all measurable. **10**

UNIT – II

2. a) Let f be defined and bounded on a measurable set of E with mE finite. In order that $\inf_{f \le \psi} \int_E \psi(x)dx = \inf_{f \ge \phi} \int_E \phi(x)dx$ for all simple functions ϕ and ψ, prove that it is necessary and sufficient that f is measurable. **10**

b) If f and g are nonnegative measurable functions, then prove that: **10**

i) $\int_E cf = c\int_E f, \ c > 0$

ii) $\int_E f + g = \int_E f + \int_E g$.

iii) If $f \le g$ a.e., then $\int_E f \le \int_E g$.

OR

c) Let f be a nonnegative function which is integrable over a set E. Then prove that for given $\in > 0$ there is $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ such that $\int_A < \in$. **10**

d) State and prove Lebesgue Convergence theorem. **10**

UNIT – III

3. a) Prove that: A function f is of bounded variation on [a, b] if and only if f is the difference of two monotone real-valued functions of [a, b]. **10**

b) If f is increasing real-valued function on [a, b], then prove that f is differentiable almost everywhere. Also prove that the derivative f' is measurable and $\int_b^a f'(x)\,dx \le f(b) - f(a)$. **10**

OR

c) If f is absolutely continuous on [a, b], then prove that it is bounded variation on [a, b]. **10**

d) If ϕ is a continuous function on (a, b) and if one derivative, say D^+ of ϕ is nondecreasing then prove that ϕ is convex. **10**

UNIT – IV

4. a) Prove that L^∞ is a normed linear space. **10**

b) State and prove Hölder inequality. **10**

OR

c) Prove that every convergent sequence is a Cauchy's sequence. **10**

d) State and prove Riesz Representation theorem. **10**

5. a) Show that: If A is countable, then $m^*A = 0$. **5**

b) Let f Be a bounded function defined on [a, b]. Prove that if f is Riemann integrable on [a, b], then it is measurable and $R\int_a^b f(x)dx = \int_a^b f(x)dx$. **5**

c) State and prove Jensen Inequality. **5**

d) Let $1 \le p \le \infty$. Then prove that for a, b, t nonnegative $(a + tb)^p \ge a^p + ptba^{(p-1)}$. **5**

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